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ON CHARACTERISTIC FUNCTIONALS OF A RANDOM VELOCITY FIELD OF HELICAL MOTIONS OF A VISCOUS FLUID

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A class of exact solutions is derived for the characteristic functional of a random velocity field of helical motions of a viscous incompressible fluid.

The solution [1, 2] of the equation for the characteristic functional of the velocity field derived in variational derivatives relates to a statistically stationary velocity field in a perfect incompressible fluid, and of the form of a Gaussian functional with constant spectral density of energy.

As first pointed out by Gromeka [3], in helical motions of a perfect incompressible fluid velocity and the velocity curl are proportional

$$\operatorname{rot} \mathbf{v} = \varkappa \mathbf{v} \tag{1}$$

where $\varkappa$ is a pseudoscalar constant whose dimension is that of the wave number. It was later shown by Steklov [4] that Gromeka's helical solutions multiplied by $\exp(-\nu\varkappa^2 t)$, where ν is the kinematic viscosity and t is the time, also satisfy the Navier-Stokes equations. Helical velocity fields in a boundless space can be either space periodic or vanishing at infinity.

The spectral form of Eq. (1) is

$$i\varepsilon_{jlm}\, k_l v_m(\mathbf{k}) = \varkappa v_j(\mathbf{k}) \qquad (i^2 = -1) \tag{2}$$

where ε_{jlm} is the Levi-Civita tensor density. The system of Eqs. (2) has a nontrivial equation for $v_j(\mathbf{k})$ only if $\mathbf{k}^2 = \varkappa^2$, which implies that the expression for the spectral density of the velocity field $v_j(\mathbf{k})$ must contain the δ-function $\delta(k - |\varkappa|)$ as a multiplier. The expression

$$v_j(k) = \left(\Delta_{jn} + i\frac{k_l}{\varkappa}\varepsilon_{jln}\right) A_n(\mathbf{k})\, \delta(k - |\varkappa|) \tag{3}$$

$$\Delta_{jn} = \delta_{jn} - k_j k_n / k^2$$

where δ_{jn} is a unit tensor, $A_n(\mathbf{k})$ is an arbitrary vector field which satisfies the condition $A_n^*(\mathbf{k}) = A_n(-\mathbf{k})$ imposed by the requirement for the velocity field to be real, satisfies system (2). Solution (3) depends on two arbitrary functions of components of vector $\mathbf{A}$ orthogonal to $\mathbf{k}$ and, consequently, is the general solution of Eq. (2) proportional to function $\delta(k - |\varkappa|)$. The spectral density of a helical velocity field $v_j(\mathbf{k}, t)$ in a viscous incompressible fluid is obtained, according to [4], by multiplying (3) by $\exp(-\nu\varkappa^2 t)$.

Let us consider the random velocity field in an unbounded fluid, whose model is the helical field. Such field may be specified by the formula for spectral density with random $\mathbf{A}(\mathbf{k})$ and $\varkappa$. The field $v_j(\mathbf{k}, t)$ in terms of field $\mathbf{A}(\mathbf{k})$ is linear, hence when field $\mathbf{A}(\mathbf{k})$ for a fixed $\varkappa$ is Gaussian, then field of v_j is also conditionally Gaussian for fixed

$\varkappa$. For this case it is possible to define the characteristic functional of the random field $v_j(\mathbf{k}, t)$. In fact, in accordance with the previously stated, the characteristic functional of field $v_j(\mathbf{k}, t)$ is the mean with respect to $\varkappa$ of some Gaussian functional

$$\varphi(\mathbf{z}, t) = \int d\sigma(\varkappa)\cdot\exp\Big\{i\int\langle v_j(\mathbf{k}, t)\rangle_\varkappa z_j\, d^3\mathbf{k} - \frac{1}{2}\int\langle v_l(\mathbf{k}_1, t)\, v_m(\mathbf{k}_2, t)\rangle_\varkappa z_l(\mathbf{k}_1)\, z_m(\mathbf{k}_2)\, d^3\mathbf{k}_1\, d^3\mathbf{k}_2\Big\} \tag{4}$$

where $\sigma(\varkappa)$ is the distribution function of parameter $\varkappa$, $\langle v_j\rangle_\varkappa$ and $\langle v_l v_m\rangle_\varkappa$ are the first and second moments of field $v_j(\mathbf{k}, t)$ relative to the conditional distribution for a fixed $\varkappa$, which by virtue of (3) are expressed in terms of moments $\langle A_j\rangle_\varkappa$ and $\langle A_j A_l\rangle_\varkappa$, and $\mathbf{z}(\mathbf{k})$ is the argument of the functional which, for example, belongs to the class of finite fields satisfying the condition $\mathbf{z}^*(\mathbf{k}) = \mathbf{z}(-\mathbf{k})$.

The characteristic functional (4) theoretically contains complete information on the random field $v_j(\mathbf{r}, t)$. Thus, for example, by substituting into the characteristic functional the expression $\exp(i\mathbf{k}\mathbf{r}_1)\,\theta_1 + \ldots + \exp(i\mathbf{k}\mathbf{r}_N)\,\theta_N$ where $\mathbf{r}_j$ and θ_j are constant vectors, for its argument, we obtain the characteristic function $F(\theta_1, \ldots, \theta_N)$ of velocity distribution of fluid particles at fixed points $\mathbf{r}_1, \ldots, \mathbf{r}_N$ [5]. Density of the multi-point distribution function of velocities $f(\mathbf{v}_1, \ldots, \mathbf{v}_N)$ can be derived from $F(\theta_1, \ldots, \theta_N)$ by the Fourier transformation

$$f(\mathbf{v}_1, \ldots, \mathbf{v}_N) = \frac{1}{(2\pi)^{3N}}\int F\exp\Big(-i\sum_{j=1}^{N}\mathbf{v}_j\theta_j\Big)\, d^3\theta_1\ldots d^3\theta_N$$

All multi-point velocity distribution functions in the considered particular case are the mean of some Gaussian functions, since the characteristic functional (4) is the mean of Gaussian functionals with respect to parameter $\varkappa$. In particular cases multi-point velocity distribution function can be Gaussian, when distribution $\sigma(\varkappa)$ is concentrated at a single point.

If field $\mathbf{A}(\mathbf{k})$ is selected so that its moments satisfy conditions

$$\begin{aligned} &\langle A_j\rangle_\varkappa \equiv 0 \\ &\langle A_l(\mathbf{k}_1) A_m(\mathbf{k}_2)\rangle_\varkappa\, \delta(k_1 - |\varkappa|)\,\delta(k_2 - |\varkappa|) = \\ &\quad f_{lm}(\mathbf{k}_2, \varkappa)\,\delta(\mathbf{k}_1 + \mathbf{k}_2)\,\delta(k_1 - |\varkappa|) \end{aligned} \tag{5}$$

then functional (4) is invariant with respect to transformation $\mathbf{z}(\mathbf{k}) \to e^{i\mathbf{k}\mathbf{a}}\mathbf{z}(\mathbf{k})$, where $\mathbf{a}$ is a constant vector, and consequently defines a statistically homogeneous field. And, if f_{lm} in formula (5) depends only on the modulus of $\mathbf{k}_2$ and $\sigma(\varkappa) = \sigma(-\varkappa)$, functional (4) is invariant also with respect to the transformation $\mathbf{z}(\mathbf{k}) \to L\mathbf{z}(L^{-1}\mathbf{k})$, where L is any arbitrary rotation or reflection of space, and defines a statistically isotropic velocity field [5].

Since helical velocity fields satisfy the linear equation (1), hence a spectral energy transfer in a random field with helical patterns is absent. However the characteristic functionals presented here are examples of exact solutions of Hopf's equations and provide a certain idea about statistical distributions of velocity fields in a viscous incompressible fluid, which is compatible with equations of hydrodynamics.

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WAVE EXCITATION IN A TWO-LAYERED MEDIUM BY AN OSCILLATING STAMP

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The antiplane problem of wave excitation in a two-layered medium consisting of an elastic layer and a rigidly connected half-space, set in oscillation by a stamp, is considered. On the basis of the physical principle of ultimate absorption [1, 2], the problem of the oscillation of a source on a surface is solved and consequently, the integral equation of a mixed problem is derived. A detailed study of the dispersion equation by using an electronic digital computer preceded the solution of the problem. The solution of the contact problem is constructed and a numerical analysis is made of the solutions obtained for specific values of the parameters.

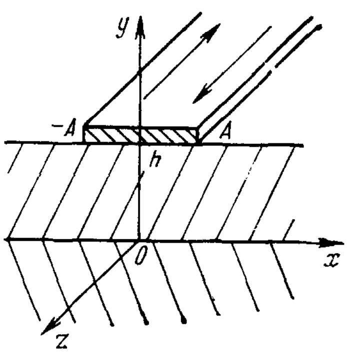

Fig. 1

1. The case is considered when forces $\tau_{xy} = \mathrm{Re}\,[\tau(x)\, e^{-i\omega t}]$ independent of the z coordinate are applied to a surface layer of thickness h in the domain $X \in [-A, A]$ (Fig. 1), and there are no normal stresses. The oscillations are assumed steady-state.

Then applying the principle of ultimate absorption by the Fourier transform method, we arrive at the following formulas describing the displacements $w(x, y, t)$ and $w_1(x, y, t)$ for the layer and the half-space, respectively:

$$w(x, y, t) = \mathrm{Re}\,[W_1(x, y)\, e^{-i\omega t}] \tag{1.1}$$

$$W(x, y) = \frac{1}{2\pi} \int_{-a}^{a} \int_{\Gamma} \frac{(\sigma + G\sigma_1)\, e^{\sigma y} + (\sigma - G\sigma_1)\, e^{-\sigma y}}{\sigma\,[(\sigma + G\sigma_1)\, e^{\sigma} - (\sigma - G\sigma_1)\, e^{-\sigma}]} \times$$

$$e^{i\alpha(\xi - x)}\, \tau(\xi)\, d\alpha\, d\xi$$